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On the invariance transformations of the energy–momentum tensor in the Einstein–Maxwell theory

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Abstract. A homothetic conformal transformation of the metric tensor is both necessary and sufficient to ensure that a source-free non-null electromagnetic field in four dimensions undergoes a duality rotation leaving its energy–momentum tensor invariant. For null electromagnetic fields the condition is sufficient but not necessary.

The result suggests that vacuum Einstein–Maxwell space–times exist which admit groups of homothetic motions. If the order of such a group is greater than unity it must be associated with isometry transformations of the metric tensor other than those obtained as a special case of the conformal group.

1. Introduction

In a four-dimensional Riemannian geometry a source-free electromagnetic field satisfying Maxwell's equations

$$F^{\mu\nu}{}_{|\nu}=0 \qquad \text{and} \qquad {}^*F^{\mu\nu}{}_{|\nu}=0, \tag{1.1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and ${}^*F^{\mu\nu}$ is its dual, has associated with it the energy--momentum tensor

$$T_{\mu\nu}=-\tfrac{1}{2}(F_{\mu\sigma}F^{\sigma}{}_{\nu}+{}^*F_{\mu\sigma}{}^*F^{\sigma}{}_{\nu}). \tag{1.2}$$

This obeys the conservation law

$$T^{\mu\nu}{}_{|\nu}=0 \tag{1.3}$$

as a consequence of Maxwell's equations (1.1).

The form of $T_{\mu\nu}$ and certain properties of antisymmetric tensors in four dimensions (see e.g. Misner and Wheeler 1957) ensures that it is invariant under transformations of the type

$$\bar{F}_{\mu\nu}=F_{\mu\nu}\cos\theta+{}^*F_{\mu\nu}\sin\theta \tag{1.4}$$

for any scalar θ (although if $F_{\mu\nu}$ is not null, $\bar{F}_{\mu\nu}$ will only represent an electromagnetic field when θ is a constant). These so called duality rotations are of significance in the general relativistic theory of the electromagnetic field since the work of Rainich (1925) and subsequently Misner and Wheeler (1957) has shown that Einstein's equations

$$R_{\mu\nu}=-8\pi T_{\mu\nu} \tag{1.5}$$

together with Maxwell's equations (1.1) require $F_{\mu\nu}$ to be determined up to a duality rotation by the metric tensor $g_{\mu\nu}$ when the two scalars

$$\alpha = F_{\mu\nu}F^{\mu\nu} \qquad \text{and} \qquad \beta = F_{\mu\nu}\,{}^*F^{\mu\nu} \tag{1.6}$$

do not simultaneously vanish (the non-null case).

Following this work it has been shown that for Einstein–Maxwell space–times which admit a group of motions the action of the transformations on the non-null electromagnetic field tensor $F_{\mu\nu}$ does indeed produce the duality rotation (1.4) as a consequence of the invariance of $R_{\mu\nu}$ (Ray and Thompson 1975, Woolley 1973a).

In the present paper we determine necessary and sufficient conditions for a group of point transformations to leave $T_{\mu\nu}$ invariant and generate a duality rotation of the non-null electromagnetic field in any four-dimensional Riemannian geometry. The conclusion is that such a group must be homothetic. For null electromagnetic fields this is not necessary. We then look at some implications of this mode of transformation.

2. The transformation of $F_{\mu\nu}$

We consider a four-dimensional Riemannian geometry with metric tensor $g_{\mu\nu}$ and assume that $g_{\mu\nu}$ admits an r-parameter group of point transformations, given infinitesimally by

$$x^{\nu} \to x^{\nu} - \epsilon^{i} \underset{i}{u}^{\nu} \tag{2.1}$$

with r independent generating vectors $\underset{i}{u}^{\nu}$ $(i = 1, 2, \ldots, r)$ which leaves invariant the energy–momentum tensor $T_{\mu\nu}$ of a non-null electromagnetic field.

The invariance of $T_{\mu\nu}$ requires that its Lie derivative $\underset{u}{£} T_{\mu\nu}$, taken with respect to any generating vector u^{ν} of the group, satisfies

$$\underset{u}{£} T_{\mu\nu} = 0 \tag{2.2}$$

and a necessary and sufficient condition for this to be true is that a coordinate system exists in which $T_{\mu\nu}$ is independent of one of the coordinates (Yano 1955, p 34).

The algebraic structure of $T_{\mu\nu}$ ensures that it obeys the Rainich identities

$$T_{\mu\sigma}T^{\sigma}{}_{\nu} = \omega^2 g_{\mu\nu}$$

where

$$4\omega^2 = T_{\mu\nu}T^{\mu\nu} \tag{2.3}$$

and $\pm\omega$ are the eigenvalues of $T_{\mu\nu}$ which only vanish everywhere in the null case.

We define the scalar ϕ by

$$\underset{u}{£}\omega = -2\phi\omega \tag{2.4}$$

and take the deformation of the metric tensor $g_{\mu\nu}$ corresponding to (2.2), to be described by

$$\underset{u}{£}\, g_{\mu\nu} = 2\phi g_{\mu\nu} + \gamma_{\mu\nu}. \tag{2.5}$$

It is now easy to deduce from (2.2) and (2.3) that $\gamma_{\mu\nu}$ must satisfy

$$\gamma^{\sigma}{}_{\sigma} = 0 \tag{2.6}$$

and

$$T_{\mu\sigma}\gamma^{\sigma}{}_{\nu}+T_{\nu\sigma}\gamma^{\sigma}{}_{\mu}=0. \tag{2.7}$$

This has the further consequence that

$$T_{\mu\nu}\gamma^{\mu\nu}=0 \tag{2.8}$$

and we see that $\gamma_{\mu\nu}$ may be interpreted as a projection operator acting on the two linear vector spaces spanned by the pairs of eigenvectors of $T_{\mu\nu}$ which belong respectively to the eigenvalues $\pm\omega$ (Woolley 1973b) for example, if v^{ν} is an eigenvector belonging to $+\omega$ then $\gamma^{\mu}{}_{\nu}v^{\nu}$ is one belonging to $-\omega$.

These results enable us to obtain the infinitesimal transformation of $F_{\mu\nu}$ by means of the identity (e.g. Woolley 1973b)

$$F_{\mu\nu}F_{\alpha\beta}+{}^*F_{\mu\nu}{}^*F_{\alpha\beta}=T_{\mu\alpha}g_{\nu\beta}-T_{\mu\beta}g_{\nu\alpha}+T_{\nu\beta}g_{\mu\alpha}-T_{\nu\alpha}g_{\mu\beta} \tag{2.9}$$

and if we Lie differentiate this with respect to u^{ν} and use the identities

$$F^{\mu\nu}\underset{u}{\pounds}F_{\mu\nu}=\tfrac{1}{2}\underset{u}{\pounds}\alpha+2\phi\alpha \qquad \text{and} \qquad {}^*F^{\mu\nu}\underset{u}{\pounds}F_{\mu\nu}=\tfrac{1}{2}\underset{u}{\pounds}\beta+2\phi\beta, \tag{2.10}$$

together with

$$\omega^2=\tfrac{1}{16}(\alpha^2+\beta^2), \tag{2.11}$$

we finally obtain the following expressions for the Lie derivatives of $F_{\mu\nu}$ and its covariant dual ${}^*F_{\mu\nu}$:

$$\underset{u}{\pounds}F_{\mu\nu}=\phi F_{\mu\nu}+\psi\,{}^*F_{\mu\nu}+\tfrac{1}{2}(F_{\mu\sigma}\gamma^{\sigma}{}_{\nu}-F_{\nu\sigma}\gamma^{\sigma}{}_{\mu}) \tag{2.12}$$

and

$$\underset{u}{\pounds}{}^*F_{\mu\nu}=\phi\,{}^*F_{\mu\nu}-\psi F_{\mu\nu}+\tfrac{1}{2}({}^*F_{\mu\sigma}\gamma^{\sigma}{}_{\nu}-{}^*F_{\nu\sigma}\gamma^{\sigma}{}_{\mu}) \tag{2.13}$$

where

$$\psi=\underset{u}{\pounds}\,\theta \qquad \text{and} \qquad \theta=\tfrac{1}{2}\tan^{-1}(\alpha/\beta). \tag{2.14}$$

Now for $F_{\mu\nu}$ to undergo a duality rotation the parameter dependence of its finite transformation requires that $\underset{u}{\pounds}F_{\mu\nu}$ is of the form

$$\underset{u}{\pounds}F_{\mu\nu}=xF_{\mu\nu}+y\,{}^*F_{\mu\nu} \tag{2.15}$$

for scalar functions x and y. Therefore $\underset{u}{\pounds}F_{\mu\nu}$ as given by (2.12) suggests that the duality rotation is not a necessary mode of transformation of $F_{\mu\nu}$ unless the term involving $\gamma_{\mu\nu}$ can be of the form of the right-hand side of (2.15). However, this latter possibility is ruled out, except in the trivial case, by the following theorem.

Theorem 2.1. A necessary and sufficient condition for $F_{\mu\nu}$ to undergo a duality rotation is that $\gamma_{\mu\nu}$ vanishes.

Proof. The sufficiency is obvious from (2.12) and (2.13). To prove the necessity we assume that $\underset{u}{\pounds}F_{\mu\nu}$ is of the form (2.15). Then the corresponding relation for $\underset{u}{\pounds}{}^*F_{\mu\nu}$ is found to be

$$\underset{u}{\pounds}{}^*F_{\mu\nu}=x\,{}^*F_{\mu\nu}-yF_{\mu\nu}+{}^*F_{\mu\sigma}\gamma^{\sigma}{}_{\nu}-{}^*F_{\nu\sigma}\gamma^{\sigma}{}_{\mu}. \tag{2.16}$$

But by virtue of (2.6) and (2.8) we find that

$$\eta_{\mu\nu}=\tfrac{1}{2}(F_{\mu\sigma}\gamma^{\sigma}{}_{\nu}-F_{\nu\sigma}\gamma^{\sigma}{}_{\mu}) \tag{2.17}$$

and its covariant dual

$$^*\eta_{\mu\nu}=\tfrac{1}{2}(^*F_{\nu\sigma}\gamma^{\sigma}{}_{\mu}-{}^*F_{\mu\sigma}\gamma^{\sigma}{}_{\nu}) \tag{2.18}$$

must satisfy

$$\eta_{\mu\nu}F^{\mu\nu}=0 \qquad \text{and} \qquad \eta_{\mu\nu}{}^*F^{\mu\nu}=0. \tag{2.19}$$

It follows that in the non-null case we must have

$$x=\phi \qquad \text{and} \qquad y=\psi. \tag{2.20}$$

This implies that $\eta_{\mu\nu}$ vanishes which further implies that $\gamma_{\mu\nu}$ commutes with $T_{\mu\nu}$:

$$T_{\mu\sigma}\gamma^{\sigma}{}_{\nu}=T_{\nu\sigma}\gamma^{\sigma}{}_{\mu}. \tag{2.21}$$

But, by (2.7), $\gamma_{\mu\nu}$ anticommutes with $T_{\mu\nu}$. Hence $\gamma_{\mu\nu}$ must vanish when $\omega\neq 0$ and the proof is complete.

However, if we bear in mind that $\gamma_{\mu\nu}$ is trace-free and

$$\underset{u}{£}g_{\mu\nu}=2\phi g_{\mu\nu}+\gamma_{\mu\nu} \tag{2.22}$$

we also have the following theorem.

Theorem 2.2. The point transformations leaving $T_{\mu\nu}$ invariant are conformal if and only if $\gamma_{\mu\nu}$ vanishes.

From these two results we conclude that an invariance transformation of $T_{\mu\nu}$ will effect a duality rotation on the non-null $F_{\mu\nu}$ if and only if it is conformal. But in this case it is not difficult to show that

$$G_{\mu\nu}=\underset{u}{£}F_{\mu\nu} \tag{2.23}$$

is a solution of Maxwell's equations (1.1) while its dual is

$$^*G^{\mu\nu}=g^{\mu\sigma}g^{\nu\tau}\underset{u}{£}{}^*F_{\sigma\tau} \tag{2.24}$$

thus the scalars ϕ and ψ must be solutions of the equations

$$F^{\mu\nu}\phi_{|\nu}+{}^*F^{\mu\nu}\psi_{|\nu}=0 \qquad \text{and} \qquad {}^*F^{\mu\nu}\phi_{|\nu}-F^{\mu\nu}\psi_{|\nu}=0 \tag{2.25}$$

and in the non-null case these require ϕ and ψ to be constant everywhere. Therefore we can state the following theorem.

Theorem 2.3. In a four-dimensional Riemannian geometry a necessary and sufficient condition for a non-null electromagnetic field tensor to undergo a duality rotation under the action of a group of invariance transformations of its energy–momentum tensor is that those are homothetic.

Since there is no obvious reason why $\gamma_{\mu\nu}$ should vanish in a general Riemannian geometry these results point to the conclusion that the duality rotation is a sufficient but not necessary mode of transformation for the non-null electromagnetic field under an invariance transformation of $T_{\mu\nu}$.

To take into account null electromagnetic fields, defined by the simultaneous vanishing of the scalars in (1.6), it is necessary to introduce a tetrad of vectors, with basis set $\{v_\mu, w_\mu, \xi_\mu, \zeta_\mu\}$ such that v_μ and w_μ are null with inner product $v^\sigma w_\sigma = 1$ while ξ_μ and ζ_μ are orthogonal unit space-like vectors satisfying $\xi^\sigma \xi_\sigma = 1$ and $\zeta^\sigma \zeta_\sigma = 1$ (with respect to a metric form of signature $+2$) with all other products vanishing. We take v^μ to lie in the principal null direction of $F_{\mu\nu}$ so that $F_{\mu\nu}$ and its covariant dual $^*F_{\mu\nu}$ can be represented by simple bivectors:

$$F_{\mu\nu} = v_\mu \zeta_\nu - v_\nu \zeta_\mu \qquad \text{and} \qquad {}^*F_{\mu\nu} = v_\mu \xi_\nu - v_\nu \xi_\mu. \tag{2.26}$$

In addition, the metric tensor $g_{\mu\nu}$ has the decomposition

$$g_{\mu\nu} = v_\mu w_\nu + v_\nu w_\mu + \xi_\mu \xi_\nu + \zeta_\mu \zeta_\nu. \tag{2.27}$$

Now it is not difficult to show that under an infinitesimal point transformation that leaves $T_{\mu\nu}$ invariant, and induces the infinitesimal duality rotation

$$\underset{u}{\pounds} F_{\mu\nu} = \phi F_{\mu\nu} + \psi\, {}^*F_{\mu\nu} \tag{2.28}$$

on the electromagnetic field, it is both necessary and sufficient that $g_{\mu\nu}$ satisfies

$$\underset{u}{\pounds}\, g_{\mu\nu} = 2\phi g_{\mu\nu} \tag{2.29}$$

i.e. the transformation must be conformal. But in this case the scalars ϕ and ψ need not be constants in order to satisfy the integrability conditions (2.25). Therefore conformal invariance transformations of the energy–momentum tensor of the null electromagnetic field need not be homothetic.

The relevance of homothetic conformal transformations in the Einstein–Maxwell theory becomes apparent when we calculate the infinitesimal change in the Ricci tensor $R_{\mu\nu}$ that such a transformation induces. For a general conformal transformation, defined by (2.29), we have

$$\underset{u}{\pounds}\, R_{\mu\nu} = 2\phi_{\mu|\nu}$$

where

$$\phi_\mu = \frac{\partial\phi}{\partial x^\mu}. \tag{2.30}$$

It follows that $R_{\mu\nu}$ is invariant under a homothetic transformation and, bearing in mind the conformal invariance of Maxwell's equations, we conclude that vacuum Einstein–Maxwell space–times ought to exist for which the electromagnetic field has the symmetry described by (2.28), with u^ν a homothetic Killing vector and ϕ and ψ constants.

We now look at some properties of fields which admit the above symmetry.

3. Homothetic invariance transformations

Consider an r-parameter group of homothetic invariance transformations of $T_{\mu\nu}$ acting on the metric tensor $g_{\mu\nu}$ and electromagnetic field tensor $F_{\mu\nu}$. For each generator $\underset{i}{u}^{\nu}$ of the group we can define a pair of constants ϕ_i and ψ_i such that

$$\underset{i}{\pounds}\, g_{\mu\nu} = 2\phi_i g_{\mu\nu} \tag{3.1}$$

and

$$\underset{i}{\pounds} F_{\mu\nu} = \phi_i F_{\mu\nu} + \psi_i {}^*F_{\mu\nu} \tag{3.2}$$

where $\underset{i}{\pounds}$ abbreviates the Lie derivative with respect to $\underset{i}{u}^{\nu}$.

To obtain the finite transformation of $F_{\mu\nu}$ corresponding to (3.2) we use the formula (Schouten 1954, p 108)

$$\bar{\Omega}^{\Lambda} = \exp(-\epsilon \underset{u}{\pounds}\, \Omega^{\Lambda}) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{\epsilon^n}{n!} \underset{u}{\pounds}^n\, \Omega^{\Lambda} \tag{3.3}$$

which gives the finite transformation $\bar{\Omega}^{\Lambda}$ of a linear differential geometric object Ω^{Λ} under the action of a one-parameter point transformation with generating vector u^{ν}. With (3.2) we obtain

$$\bar{F}_{\mu\nu} = \mathrm{e}^{-\epsilon\phi_i}[F_{\mu\nu} \cos(\epsilon\psi_i) - {}^*F_{\mu\nu} \sin(\epsilon\psi_i)]. \tag{3.4}$$

In an Einstein–Maxwell space–time (3.4) could be interpreted as showing that a test observer moving along a trajectory of the congruence determined by $\underset{i}{u}^{\nu}$ would measure both a change in amplitude and rotation of the local electric and magnetic field vectors.

As a consequence of (3.2) and the commutation relations (Yano 1955, p 29)

$$\underset{i}{\pounds}\underset{j}{\pounds} - \underset{j}{\pounds}\underset{i}{\pounds} = c_{ij}^{k} \underset{k}{\pounds}, \tag{3.5}$$

where the c_{ij}^{k} are the group structure constants, it is found that the two sets $\{\phi_i | i = 1, 2, \ldots, r\}$ and $\{\psi_i | i = 1, 2, \ldots, r\}$ of constants which characterise the transformation are determined by the group structure through

$$c_{ij}^{k}\phi_k = 0 \qquad \text{and} \qquad c_{ij}^{k}\psi_k = 0. \tag{3.6}$$

When the group of homothetic motions is non-trivial, in the sense that not all of the ϕ_i are zero, we can regard the set of constants $\{\phi_i\}$ as representing a vector in an abstract linear space of dimension r. Then, by a well known theorem in linear algebra, we can find $r-1$ independent vectors of constant components $\underset{i}{a}^{j}$ such that

$$\underset{i}{a}^{j}\phi_j = 0 \qquad (i = 1, 2, \ldots, r-1) \tag{3.7}$$

when summation is over the order of the group. It follows from the linear character of the Lie derivative that the vectors

$$\underset{i}{v}^{\nu} = \underset{i}{a}^{j}\underset{j}{u}^{\nu} \tag{3.8}$$

generate motions. Therefore we can state the following theorem.

Theorem 3.1. A group of homothetic transformations of order greater than unity must be associated with invariance transformations of the metric tensor.

The significance of this result lies in the fact that, apart from a special case when $r=2$, the constants $\underset{i}{a}^{j}$ can be chosen so that the Killing vector defined by (3.8) also generates an invariance transformation of $F_{\mu\nu}$. To see this it is only necessary to complement the abstract vector ϕ_i with that represented by the set of components $\{\psi_i\}$. Then, depending on whether or not the two vectors ϕ_i and ψ_i are independent, we can find at least $r-2$ independent sets of constants $\underset{i}{b}^{j}$ such that

$$\underset{i}{b}^{j}\phi_j=0 \qquad \text{and} \qquad \underset{i}{b}^{\delta}\psi_j=0. \tag{3.9}$$

An exceptional case occurs when $r=2$ and the vectors ϕ_i and ψ_i are linearly independent. Then (as a referee has been kind enough to point out) no non-zero $\underset{i}{b}^{j}$ can satisfy (3.9). It follows that when $r>2$ the Killing vectors

$$\underset{i}{v}^{\nu}=\underset{i}{b}^{j}\underset{j}{u}^{\nu} \tag{3.10}$$

generate invariance transformations of $F_{\mu\nu}$. Therefore corresponding to theorem 3.1 we have the following theorem.

Theorem 3.2. If the energy–momentum tensor of a source-free electromagnetic field admits a group of homothetic invariance transformations of order greater than two the electromagnetic field tensor must itself be invariant under the transformations generated by one or more Killing vectors.

A useful aspect of this result is that it effectively imposes conditions on the components of the source-free electromagnetic field tensor when this undergoes a duality rotation generated by a number (generally greater than two) of homothetic Killing vectors. The conditions take the forms

$$F_{\mu\nu}v^{\nu}=\frac{\partial\chi}{\partial x^{\mu}} \qquad \text{and} \qquad {}^{*}F_{\mu\nu}v^{\nu}=\frac{\partial\eta}{\partial x^{\mu}} \tag{3.11}$$

where χ and η are locally defined scalar potential functions and v^{ν} is a Killing vector determined by (3.10). The pair of functions χ and η have their functional forms determined by (3.2) and the group structure.

4. Conclusions

The invariance transformations of the energy–momentum tensor which affect a duality rotation on the source-free electromagnetic field are restricted to those of the conformal group.

The fact that such transformations must generally be homothetic leads to the conclusion that vacuum Einstein–Maxwell space–times ought to exist which admit groups of homothetic motions. It might be that such space–times are of relevance in the theory of singularities, since we have seen that the amplitude of the local electric and magnetic field vectors must increase (or decrease) exponentially for an observer moving in the direction of the conformal Killing vector.

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